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A note on enhanced (G'/G)-expansion method in nonlinear physics

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Abstract In this talk we have applied an enhanced (G'/G)-expansion method to find the traveling wave solutions of the (2 + 1)-dimensional Zoomeron equation. The efficiency of this method for finding the exact solutions has been demonstrated. As a result, a set of exact solutions are derived, which can be expressed by the hyperbolic and trigonometric functions involving several parameters. When these parameters are taken as special values, the solitary wave solutions and the periodic wave solutions have been originated from the exact solutions. It has been shown that this method is effective and can be used for many other nonlinear evolution equations (NLEEs) in mathematical physics.

1. Introduction

NLEEs are encountered in various fields of mathematics, physics, chemistry, biology, engineering and numerous applications. Exact solutions of NLEEs play an important role in the proper understanding of qualitative features of many phenomena and processes in various areas of natural science. Exact solutions of nonlinear equations graphically demonstrate and allow unscrambling the mechanisms of many complex nonlinear phenomena such as spatial localization of transfer processes, multiplicity or absence steady states under various conditions, existence of peaking regimes and many others. Even those special exact solutions that do not have a clear physical meaning can be used as test problems to verify the consistency and estimate errors of various numerical, asymptotic, and approximate analytical methods. Exact solutions can serve as a basis for perfecting and testing computer algebra software packages for solving NLEEs. It is significant that many equations of physics, chemistry, and biology contain empirical parameters or empirical functions. Exact solutions

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allow researchers to design and run experiments, by creating appropriate natural conditions, to determine these parameters or functions. Therefore, investigation of exact traveling wave solutions is becoming successively attractive in nonlinear sciences day by day. However, not all equations posed of these models are solvable. As a result, many new techniques have been successfully developed by diverse groups of mathematicians and physicists, such as, the Hirota's bilinear transformation method [1,2], the Modified simple equation method [3–5], the tanh-function method [6], the Exp-function method [7–10], the Jacobi elliptic function method [11], the (G'/G)-expansion method [12–22], the homotopy perturbation method [23–25], the enhanced (G'/G)-expansion method [26,27], the Kudryashov method [28], and the tanh–coth function method [29,30].

Various ansatze have been proposed for seeking traveling wave solutions of nonlinear differential equations. The choice of an appropriate ansätze is of great importance in the direct methods.

Recently, Wang et al. [14] have introduced a simple method which is called the (G'/G)-expansion method to look for traveling wave solutions of nonlinear evolution equations, where $G = G(\xi)$ satisfies the second order linear ordinary differential equation $G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0$, where λ and μ are arbitrary constants and $u(\xi) = \alpha_m \left(\frac{G'}{G}\right)^m + \ldots$ be the traveling wave solution of NLEEs. By means of this method they have solved the KdV equation, the mKdV equation, the variant Boussinesq equations and the Hirota–Satsuma equations.

Guo and Zhou [21] have introduced an another method so called extended (G'/G)-expansion method where $G = G(\xi)$ satisfies the second order linear ordinary differential equation:

$G'' + \mu G = 0$, where $G' = \frac{dG(\xi)}{d\xi}, G'' = \frac{d^2G(\xi)}{d\xi^2}$, $\xi = x - Vt$, V is a constant and $u(\xi) = a_0 + \sum_{i=1}^{n}(a_i(G'/G)^i + b_i(G'/G)^{i-1}\sqrt{\sigma\left(1 + \frac{(G'/G)^2}{\mu}\right)})$ be the traveling wave solution. They proposed extended (G'/G)-expansion method to construct traveling wave solutions of Whitham–Broer–Kaup–Like equations and coupled Hirota–Satsuma KdV equations.

For further references of the (G'/G)-expansion method see the articles [12–22].

Among those approaches, an enhanced (G'/G)-expansion method is a tool to reveal the solitons and periodic wave solutions of NLEEs in mathematical physics and engineering. The main ideas of the enhanced (G'/G)-expansion method are that the traveling wave solutions of NLEEs can be expressed as rational functions of (G'/G), where $G = G(\xi)$ satisfies the second order linear ordinary differential equation $G'' + \mu G = 0$. The main advantage of this method is that new exact solutions of many nonlinear evolution equations can be determine more successfully in comparison with other methods.

The objective of this article is to present an enhanced (G'/G)-expansion method to construct the exact solutions for NLEEs in mathematical physics via the (2 + 1)-dimensional Zoomeron equation. The Zoomeron equation is completely integrable. Therefore, it has N-soliton solutions.

The article is arranged as follows: In Section 2, the enhanced (G'/G)-expansion method is discussed. In Section 3, we apply this method to the nonlinear evolution equations pointed out above; in Section 4, results and discussions; in Section 5, comparisons, and in Section 6 conclusions are given.

2. An enhanced (G'/G)-expansion method

In this section, we describe the enhanced (G'/G)-expansion method for finding traveling wave solutions of NLEEs. Suppose that a nonlinear partial differential equation, say in two independent variables x and t is given by

$$\Re(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, \ldots) = 0, \tag{2.1}$$

where $u(\xi) = u(x, t)$ is an unknown function, $\Re$ is a polynomial of $u(x, t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. In the following, we give the main steps of this method [26,27]:

Step 1. Combining the independent variables x and t into one variable $\xi = x \pm \omega t$, we suppose that

$$u(\xi) = u(x, t), \qquad \xi = x \pm \omega t. \tag{2.2}$$

The traveling wave transformation Eq. (2.2) permits us to reduce Eq. (2.1) to the following ODE:

$$\Re(u, u', u'', \ldots) = 0, \tag{2.3}$$

where $\Re$ is a polynomial in $u(\xi)$ and its derivatives, while $u'(\xi) = \frac{du}{d\xi}, u''(\xi) = \frac{d^2u}{d\xi^2}$ and so on.

Step 2. We suppose that Eq. (2.3) has the formal solution

$$u(\xi) = \sum_{i=-n}^{n}\left(\frac{a_i(G'/G)^i}{(1+\lambda(G'/G))^i} + b_i(G'/G)^{i-1}\sqrt{\sigma\left(1+\frac{(G'/G)^2}{\mu}\right)}\right), \tag{2.4}$$

where $G = G(\xi)$ satisfies the equation $G'' + \mu G = 0$, (2.5)

in which $a_i, b_i(-n \leqslant i \leqslant n; n \in \mathrm{N})$ and λ are constants to be determined later, and $\sigma = \pm 1, \mu \neq 0$.

Step 3. The positive integer n can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in Eq. (2.1) or Eq. (2.3). Moreover precisely, we define the degree of $u(\xi)$ as $D(u(\xi)) = n$ which gives rise to the degree of other expression as follows:

$$D\left(\frac{d^q u}{d\xi^q}\right) = n + q, D\left(u^p\left(\frac{d^q u}{d\xi^q}\right)^s\right) = np + s(n+q). \tag{2.6}$$

Therefore we can find the value of n in Eq. (2.4), using Eq. (2.6).

Step 4. We substitute Eq. (2.4) into Eq. (2.3) using Eq. (2.5) and then collect all terms of same powers of $(G'/G)^j$ and $(G'/G)^j\sqrt{\sigma\left(1+\frac{1}{\mu}(G'/G)^2\right)}$ together, then set each coefficient of them to zero to yield a over-determined system of algebraic equations, solve this system for a_i, b_i, λ and ω.

Step 5. From the general solution of Eq. (2.5), we get

When $\mu < 0$,

$$\frac{G'}{G} = \sqrt{-\mu}\tanh(A + \sqrt{-\mu}\xi) \tag{2.7}$$

$$\text{And } \frac{G'}{G} = \sqrt{-\mu}\coth(A + \sqrt{-\mu}\xi) \tag{2.8}$$

Again, when $\mu > 0$,

$$\frac{G'}{G} = \sqrt{\mu}\tan(A - \sqrt{\mu}\xi) \tag{2.9}$$

$$\text{And}\ \frac{G'}{G} = \sqrt{\mu}\cot(A + \sqrt{\mu}\xi) \tag{2.10}$$

where A is an arbitrary constant. Finally, substituting $a_i, b_i(-n \leqslant i \leqslant n; n \in \mathrm{N}), \lambda, \omega$ and Eqs. (2.7)–(2.10) into Eq. (2.4) we obtain traveling wave solutions of Eq. (2.1).

3. Application

In this section, we will exert enhanced (G'/G)-expansion method to solve the (2 + 1)-dimensional Zoomeron equation in the form,

$$\left(\frac{u_{xy}}{u}\right)_{tt} - \left(\frac{u_{xy}}{u}\right)_{xx} + 2(u^2)_{xt} = 0, \tag{3.1}$$

where $u(x, y, t)$ is the amplitude of the relative wave mode. This equation is one of incognito evolution equation. The equation was introduced by Calogero and Degasperis [31]. In the literature, there are a few works about this equation. Recently, Abazari [22] obtained periodic and soliton solutions to Zoomeron equation by means of (G'/G)-expansion method.

The traveling wave transformation equation $u(x, y, t) = u(\xi)$, $\xi = x + y - \omega t$ transform Eq. (3.1) to the following ordinary differential equation:

$$\omega^2\left(\frac{u''}{u}\right)'' - \left(\frac{u''}{u}\right)'' - 2\omega(u^2)'' = 0. \tag{3.2}$$

Now integrating Eq. (3.2) with respect to ξ twice, we have

$$(\omega^2 - 1)u'' - 2\omega u^3 + ku = 0, \tag{3.3}$$

where k is a constant of integration. Balancing the highest-order derivative term u'' and the nonlinear term u^3 from Eq. (3.3), yields $3n = n + 2$ which gives $n = 1$.

Hence for $n = 1$ Eq. (2.4) reduces to

$$\begin{aligned} u(\xi) = {} & \frac{a_{-1}(1 + \lambda(G'/G))}{(G'/G)} + a_0 + \frac{a_1(G'/G)}{1 + \lambda(G'/G)} \\ & + b_{-1}(G'/G)^{-2}\sqrt{\sigma\left(1 + \frac{1}{\mu}(G'/G)^2\right)} \\ & + b_0(G'/G)^{-1}\sqrt{\sigma\left(1 + \frac{1}{\mu}(G'/G)^2\right)} \\ & + b_1\sqrt{\sigma\left(1 + \frac{1}{\mu}(G'/G)^2\right)}, \end{aligned} \tag{3.4}$$

where $G = G(\xi)$ satisfies Eq. (2.5). Substitute Eq. (3.4) along with Eq. (2.5) into Eq. (3.3). As a result of this substitution, we get a polynomial of $(G'/G)^j$ and $(G'/G)^j\sqrt{\sigma\left(1 + \frac{1}{\mu}(G'/G)^2\right)}$. From these polynomials, we equate the coefficients of $(G'/G)^j$ and $(G'/G)^j\sqrt{\sigma\left(1 + \frac{1}{\mu}(G'/G)^2\right)}$, and setting them to zero, we get an over-determined system that consists of twenty-five algebraic equations. Solving this system for a_i, b_i, λ and ω, we obtain the following sets:

Set 1 : $k = -2\mu(\omega^2 - 1), \omega = \omega, \lambda = 0, a_{-1} = 0, a_0 = 0,$

$a_1 = \pm\sqrt{\left(\frac{\omega^2 - 1}{\omega}\right)}, b_{-1} = 0, b_0 = 0, b_1 = 0.$

$k = -2\mu(\omega^2 - 1), \omega = \omega, \lambda = 0, a_{-1} = \pm\mu\sqrt{\left(\frac{\omega^2 - 1}{\omega}\right)},$

$a_0 = 0, a_1 = 0, b_{-1} = 0, b_0 = 0, b_1 = 0.$

$k = -2\mu(\omega^2 - 1), \omega = \omega, \lambda = \lambda, a_{-1} = \pm\mu\sqrt{\left(\frac{\omega^2 - 1}{\omega}\right)},$

$a_0 = \mp\mu\lambda\sqrt{\left(\frac{\omega^2 - 1}{\omega}\right)}, a_1 = 0, b_{-1} = 0, b_0 = 0, b_1 = 0.$

Set 2 : $k = -2\mu(\omega^2 - 1) + 6\mu\left(\frac{\omega^2 - 1}{\omega}\right), \omega = \omega, \lambda = 0,$

$a_{-1} = \pm\mu\sqrt{\left(\frac{\omega^2 - 1}{\omega}\right)}, a_0 = 0, a_1 = \pm\sqrt{\left(\frac{\omega^2 - 1}{\omega}\right)},$

$b_{-1} = 0, b_0 = 0, b_1 = 0.$

Set 3 : $k = \mu(\omega^2 - 1), \omega = \omega, \lambda = \lambda, a_{-1} = 0, a_0 = 0, a_1 = 0,$

$b_{-1} = 0, b_0 = \pm\mu\sqrt{\left(\frac{\omega^2 - 1}{\omega}\right)}, b_1 = 0.$

$k = \mu(\omega^2 - 1), \omega = \omega, \lambda = \lambda, a_{-1} = 0, a_0 = 0, a_1 = 0, b_{-1} = 0,$

$b_0 = 0, b_1 = \pm\sqrt{\left(\frac{(\omega^2 - 1)\mu}{\omega\sigma}\right)}.$

Set 4 : $k = -\frac{1}{2}\mu(\omega^2 - 1), \omega = \omega, \lambda = 0, a_{-1} = 0, a_0 = 0,$

$a_1 = \pm\frac{1}{2}\sqrt{\left(\frac{\omega^2 - 1}{\omega}\right)}, b_{-1} = 0, b_0 = 0,$

$b_1 = \pm\frac{1}{2}\sqrt{\left(\frac{(\omega^2 - 1)\mu}{\omega\sigma}\right)}.$

$k = -\frac{1}{2}\mu(\omega^2 - 1), \omega = \omega, \lambda = \lambda, a_{-1} = \pm\frac{\mu}{2}\sqrt{\left(\frac{(\omega^2 - 1)}{\omega}\right)},$

$a_0 = \mp\frac{\mu\lambda}{2}\sqrt{\left(\frac{(\omega^2 - 1)}{\omega}\right)}, a_1 = 0, b_{-1} = 0,$

$b_0 = \pm\frac{\mu}{2}\sqrt{\left(\frac{(\omega^2 - 1)}{\omega\sigma}\right)}, b_1 = 0.$

Now substituting Sets 1–4 and Eq. (2.5) into Eq. (3.4), we deduce copious traveling wave solutions of Eq. (3.1) respectively as follows.

When $\mu < 0$ and $\xi = x + y - \omega t$, we get the following hyperbolic function solutions:

Family 1. $u_{1,2}(\xi) = \pm\sqrt{-\mu}\sqrt{\left(\frac{\omega^2 - 1}{\omega}\right)}\tanh(A + \sqrt{-\mu}\xi),$

$u_{3,4}(\xi) = \pm\sqrt{-\mu}\sqrt{\left(\frac{\omega^2 - 1}{\omega}\right)}\coth(A + \sqrt{-\mu}\xi),$

Family 2. $u_{5,6}(\xi) = \pm 2\sqrt{-\mu}\sqrt{\left(\frac{\omega^2-1}{\omega}\right)}\operatorname{csch}(2(A+\sqrt{-\mu}\xi)),$

Family 3. $u_{7,8}(\xi) = \mp\sqrt{-\mu}\sqrt{\left(\frac{(\omega^2-1)}{\omega}\right)}\operatorname{csch}(A+\sqrt{-\mu}\xi),$

$$u_{9,10}(\xi) = \mp I\sqrt{-\mu}\sqrt{\left(\frac{(\omega^2-1)}{\omega}\right)}\operatorname{sech}(A+\sqrt{-\mu}\xi),$$

Family 4. $u_{11,12}(\xi) = \pm\frac{1}{2}\sqrt{-\mu}\sqrt{\left(\frac{(\omega^2-1)}{\omega}\right)}(\tanh(A+\sqrt{-\mu}\xi) + I\operatorname{sech}(A+\sqrt{-\mu}\xi)),$

$$u_{13,14}(\xi) = \pm\frac{1}{2}\sqrt{-\mu}\sqrt{\left(\frac{(\omega^2-1)}{\omega}\right)}(\coth(A+\sqrt{-\mu}\xi) + \operatorname{csch}(A+\sqrt{-\mu}\xi)),$$

Consequently, When $\mu > 0$ and $\xi = x + y - \omega t$, we obtain the following plane periodic solutions:

Family 5. $u_{15,16}(\xi) = \pm\sqrt{\mu}\sqrt{\left(\frac{\omega^2-1}{\omega}\right)}\tan(A-\sqrt{\mu}\xi),$

$$u_{17,18}(\xi) = \pm\sqrt{\mu}\sqrt{\left(\frac{\omega^2-1}{\omega}\right)}\cot(A+\sqrt{\mu}\xi),$$

Family 6. $u_{19,20}(\xi) = \pm 2\sqrt{\mu}\sqrt{\left(\frac{\omega^2-1}{\omega}\right)}\csc(2(A\mp\sqrt{\mu}\xi)),$

Family 7. $u_{21,22}(\xi) = \mp\sqrt{\mu}\sqrt{\left(\frac{(\omega^2-1)}{\omega}\right)}\csc(A-\sqrt{\mu}\xi),$

$$u_{23,24}(\xi) = \mp\sqrt{\mu}\sqrt{\left(\frac{(\omega^2-1)}{\omega}\right)}\sec(A+\sqrt{\mu}\xi),$$

Family 8. $u_{25,26}(\xi) = \pm\frac{1}{2}\sqrt{\mu}\sqrt{\left(\frac{(\omega^2-1)}{\omega}\right)}(\tan(A-\sqrt{\mu}\xi) + I\sec(A-\sqrt{\mu}\xi)),$

$$u_{27,28}(\xi) = \pm\frac{1}{2}\sqrt{\mu}\sqrt{\left(\frac{(\omega^2-1)}{\omega}\right)}(\cot(A+\sqrt{\mu}\xi) + \csc(A+\sqrt{\mu}\xi)).$$

Remark: We have checked all the obtained solutions by putting them back into the original equation and found correct. From the obtained solutions we observe that $\omega \neq 0, \pm 1$.

4. Results and discussion

In this section we will discuss about the desired solutions of (2 + 1)-dimensional Zoomeron equation. It is interesting to point out that the delicate balance between the nonlinearity effect and the linear effect gives rise to solitons, that after a fully interaction with others, the solitons come back retaining their identities with the same speed and shape. If two solitons collide, then these just pass through each other and emerge unchanged.

The determined solutions from Family 1 to Family 4, for $\mu < 0$, are hyperbolic function solutions which are traveling wave solutions. For $\mu = -1$, $A = y = 0$ and wave speed $\omega = 2$, Family 1 ($u_{1,2}(\xi)$) are kink wave solution within the interval $-3 \leqslant x, t \leqslant 3$ represented in Fig. 1. Fig. 2 represents singular kink wave solutions for $\mu = -1$, $A = y = 0$ and wave speed $\omega = 2$ within the interval $-3 \leqslant x, t \leqslant 3$ (only shows the shape of $u_{5,6}(\xi)$). Fig. 3 represents Bell shaped solition of $u_{9,10}(\xi)$ for the values of $\mu = -1$, $A = y = 0$ and wave speed $\omega = 2$ within the interval $-3 \leqslant x, t \leqslant 3$. For the values of $\mu = -5$, $A = y = 0$ and wave speed $\omega = -7$ within the interval $-3 \leqslant x, t \leqslant 3$, $u_{13,14}(\xi)$ are singular soliton solutions represented in Fig. 4.

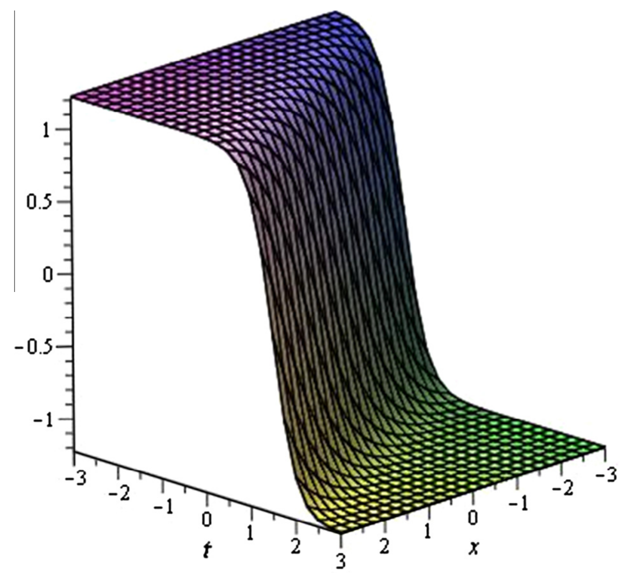

Figure 1 Kink shape of ($u_{1,2}(\xi)$) for $\mu = -1$, $A = y = 0$ and $\omega = 2$.

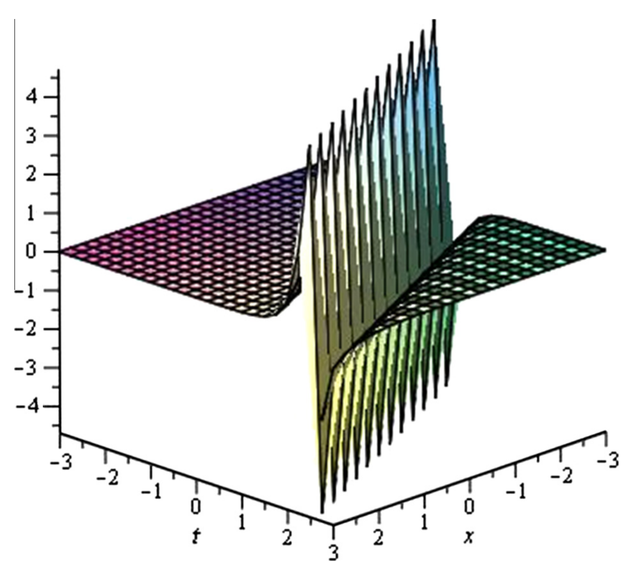

Figure 2 Singular kink shape of $u_{5,6}(\xi)$ for $\mu = -1$, $A = y = 0$ and $\omega = 2$.

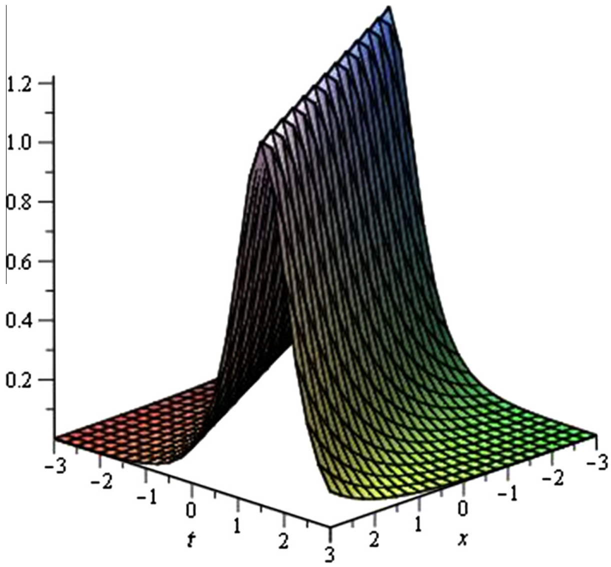

Figure 3 Bell shaped profile of $u_{9,10}(\xi)$ for $\mu = -1$, $A = y = 0$ and $\omega = 2$.

Consequently, for $\mu > 0$, Family 5–Family 8 are trigonometric function solutions, also said to be plane periodic traveling wave solutions are represented in Figs. 5–8 respectively.

The wave speed ω plays an important role in the physical structure of the solutions obtained above. For the positive values of wave speed ω the disturbance represented by $u(\xi) = u(x - \omega t)$ are moving in the positive x-direction. Consequently, the negative values of wave speed ω the disturbance represented by $u(\xi) = u(x - \omega t)$ are moving in the negative x-direction.

4.1. Graphical representation

Some of our obtained traveling wave solutions are represented in the figures with the aid of commercial software Maple:

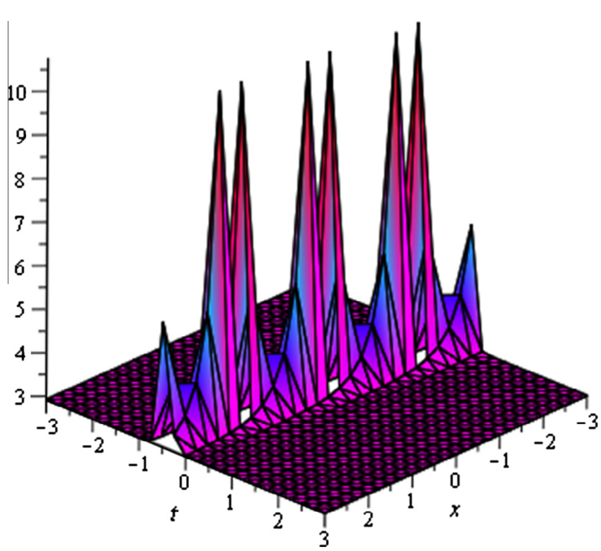

Figure 4 Singular soliton of $u_{13,14}(\xi)$For $\mu = -5$, $A = y = 0$ and $\omega = -7$.

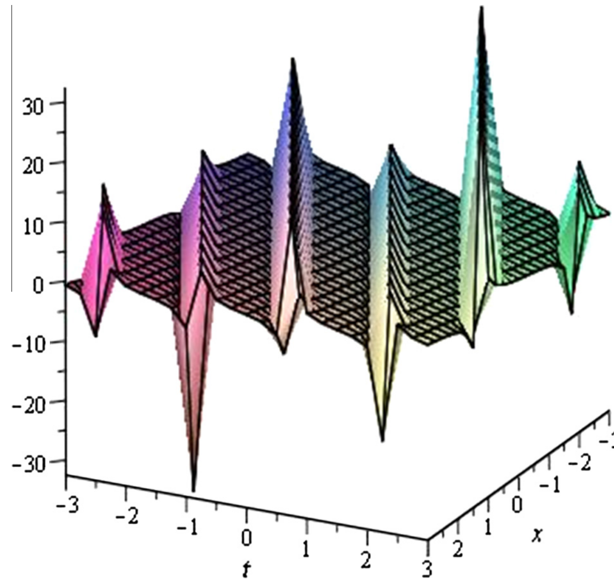

Figure 5 Periodic profile of $u_{15,16}(\xi)$ for $\mu = 1$, $A = y = 0$ and $\omega = 2$.

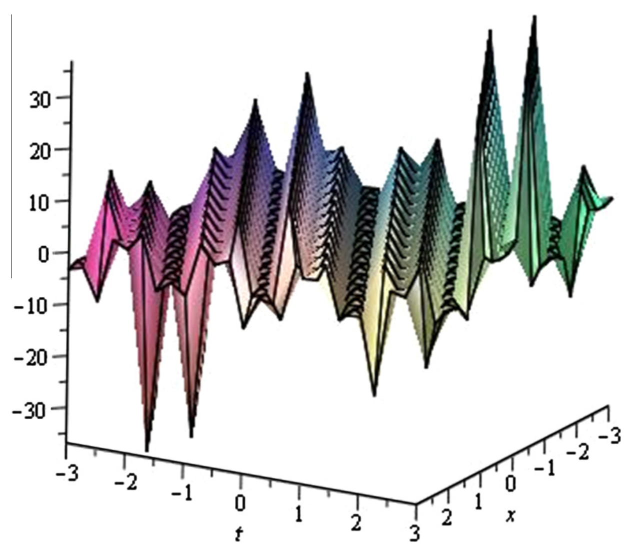

Figure 6 Periodic profile of $u_{19,20}(\xi)$ for $\mu = 1$, $A = y = 0$ and $\omega = 2$.

5. Comparisons

Comparison with (G'/G)-expansion method: Abazari [22] examined exact solutions of the (2 + 1)-dimensional Zoomeron equation by using the (G'/G)-expansion method and obtained five solutions (see Appendix A). On the contrary by using the enhanced (G'/G)-expansion method in this article we have obtained fourteen solutions. It is remarkable to point out that for particular values of the parameters some of our solutions obtained by enhanced (G'/G)-expansion method are coincided with existing solutions of Abazari [22] which were obtained by (G'/G)-expansion method. The comparisons among the solutions of (G'/G)-expansion method done by

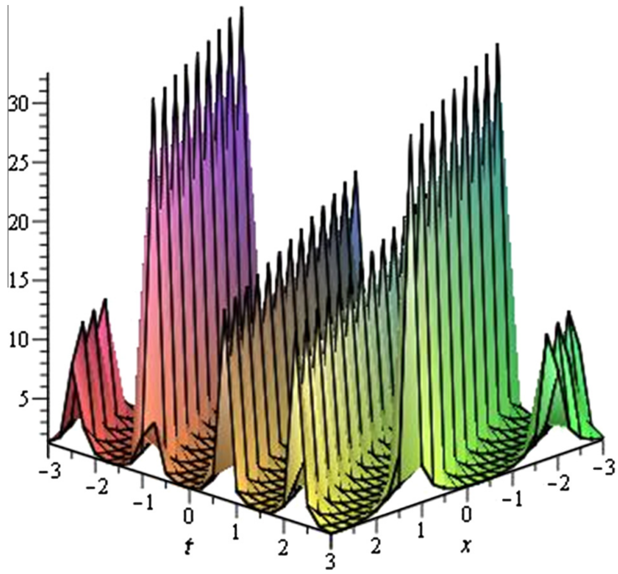

Figure 7 Periodic profile of $u_{23,24}(\xi)$ for $\mu = 1$, $A = y = 0$ and $\omega = 2$.

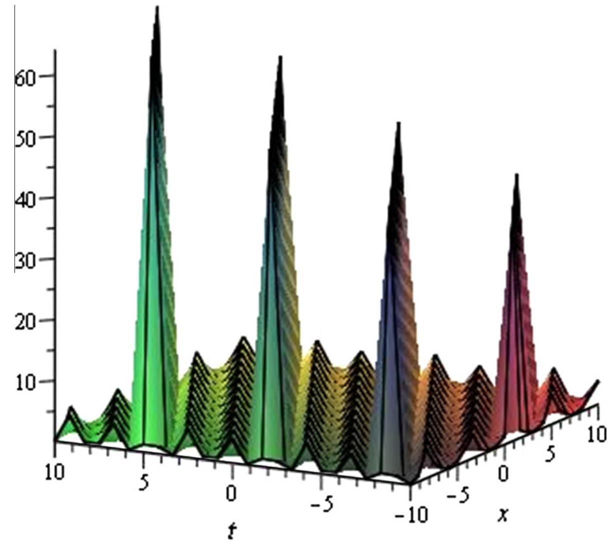

Figure 8 Periodic profile of $u_{27,28}(\xi)$ for $\mu = 2$, $A = 2$, $y = 0$ and $\omega = 2$.

Abazari [22] and the enhanced (G'/G)-expansion method used in the article are shown in the following table:

Solutions of Abazari [22] obtained by (G'/G)-expansion method	Solutions obtained by enhanced (G'/G)-expansion method in this article
i. If we set $c = -1$ and $\eta_H = 0$ then the solution (21a) becomes $u_H(\xi) = \mp\frac{1}{2}\sqrt{\left(\frac{-2R}{\omega}\right)}\tanh\left(-\frac{1}{2}\sqrt{\left(\frac{-2R}{\omega^2-1}\right)}\xi\right)$, where $\xi = x + y - \omega t$	i. If we set $\sqrt{-\mu} = -\frac{1}{2}\sqrt{\left(\frac{-2R}{\omega^2-1}\right)}$ and $A = 0$ in our solution $u_{1,2}(\xi)$ then it becomes $u_{1,2}(\xi) = \mp\frac{1}{2}\sqrt{\left(\frac{-2R}{\omega}\right)}\tanh\left(-\frac{1}{2}\sqrt{\left(\frac{-2R}{\omega^2-1}\right)}\xi\right)$, where $\xi = x + y - \omega t$
ii. If we set $c = -1$ and $\eta_H = 0$ then the solution (21 b) becomes $u_H(\xi) = \mp\frac{1}{2}\sqrt{\left(\frac{-2R}{\omega}\right)}\coth\left(-\frac{1}{2}\sqrt{\left(\frac{-2R}{\omega^2-1}\right)}\xi\right)$, where $\xi = x + y - \omega t$	ii. If we set $\sqrt{-\mu} = -\frac{1}{2}\sqrt{\left(\frac{-2R}{\omega^2-1}\right)}$ and $A = 0$ in our solution $u_{3,4}(\xi)$ then it becomes $u_{3,4}(\xi) = \mp\frac{1}{2}\sqrt{\left(\frac{-2R}{\omega}\right)}\coth\left(-\frac{1}{2}\sqrt{\left(\frac{-2R}{\omega^2-1}\right)}\xi\right)$, where $\xi = x + y - \omega t$
iii. If we set $c = -1$, $R = -R$ and $\eta_T = 0$ then the solution (23a) becomes $u_T(\xi) = \mp\frac{1}{2}\sqrt{\left(\frac{2R}{\omega}\right)}\tan\left(-\frac{1}{2}\sqrt{\left(\frac{2R}{\omega^2-1}\right)}\xi\right)$, where $\xi = x + y - \omega t$	iii. If we set $\sqrt{\mu} = \frac{1}{2}\sqrt{\left(\frac{2R}{\omega^2-1}\right)}$, and $A = 0$ in our solution $u_{15,16}(\xi)$ then it becomes $u_{15,16}(\xi) = \mp\frac{1}{2}\sqrt{\left(\frac{2R}{\omega}\right)}\tan\left(-\frac{1}{2}\sqrt{\left(\frac{2R}{\omega^2-1}\right)}\xi\right)$, where $\xi = x + y - \omega t$
iv. i. If we set $c = -1$, $R = -R$ and $\eta_T = 0$ then the solution (23b) becomes $u_T(\xi) = \mp\frac{1}{2}\sqrt{\left(\frac{2R}{\omega}\right)}\cot\left(-\frac{1}{2}\sqrt{\left(\frac{2R}{\omega^2-1}\right)}\xi\right)$, where $\xi = x + y - \omega t$	iv. If we set $\sqrt{\mu} = -\frac{1}{2}\sqrt{\left(\frac{2R}{\omega^2-1}\right)}$, and $A = 0$ in our solution $u_{17,18}(\xi)$ then it becomes$u_{17,18}(\xi) = \mp\frac{1}{2}\sqrt{\left(\frac{2R}{\omega}\right)}\cot\left(-\frac{1}{2}\sqrt{\left(\frac{2R}{\omega^2-1}\right)}\xi\right)$, where $\xi = x + y - \omega t$

6. Conclusions

In this paper, an enhanced (G'/G)-expansion method has been successfully applied to find the solitary wave solutions for the (2 + 1)-dimensional Zoomeron equation. The method has been used to find new exact solutions. As a result, hyperbolic function solutions, and trigonometric function solutions with several free parameters have been obtained. The obtained solutions with free parameters may be important to explain physical phenomena. The paper shows that the devised algorithm is effective and can be used for many other NLEEs in mathematical physics. Thus, we can say that the enhanced (G'/G)-expansion method can be extended to solve the problems of nonlinear partial differential equations which arising in the theory of soliton and other nonlinear sciences.

Appendix A.

By using the (G'/G)-expansion method Abazari [22] obtained the following three types of traveling wave solutions:

Case 1. For $\lambda^2 - 4\mu > 0$,

$$u_H(x,y,t) = \mp\frac{1}{2}\sqrt{\left(\frac{-2R}{\omega}\right)} \times \tanh\left(-\frac{1}{2}\sqrt{\left(\frac{2R}{\omega^2-1}\right)}(x - cy - \omega t) - \eta_H\right), \tag{21a}$$

$$u_H(x,y,t) = \mp\frac{1}{2}\sqrt{\left(\frac{-2R}{\omega}\right)} \times \coth\left(-\frac{1}{2}\sqrt{\left(\frac{2R}{\omega^2-1}\right)}(x-cy-\omega t)-\eta_H\right), \tag{21b}$$

where $\eta_H = \tanh^{-1}\left(\frac{C_1}{C_2}\right)$, $C_1^2 < C_2^2$ and C_1, C_2 are arbitrary constants.

Case 2. For $\lambda^2 - 4\mu < 0$,

$$u_T(x,y,t) = \mp\frac{1}{2}\sqrt{\left(\frac{-2R}{\omega}\right)} \times \tan\left(-\frac{1}{2}\sqrt{\left(\frac{2R}{\omega^2-1}\right)}(x-cy-\omega t)-\eta_T\right), \tag{23a}$$

$$u_T(x,y,t) = \mp\frac{1}{2}\sqrt{\left(\frac{-2R}{\omega}\right)} \times \cot\left(-\frac{1}{2}\sqrt{\left(\frac{2R}{\omega^2-1}\right)}(x-cy-\omega t)-\eta_T\right), \tag{23b}$$

where $\eta_T = \tan^{-1}\left(\frac{C_1}{C_2}\right)$, $C_1^2 > C_2^2$ and C_1, C_2 are arbitrary constants.

Case 3. For $\lambda^2 - 4\mu = 0$,

$$u_{rat}(x,y,t) = \mp\frac{c(\omega^2-1)C_2}{\omega(C_1+C_2(x-cy-\omega t))\sqrt{\left(-\frac{c(\omega^2-1)}{\omega}\right)}}. \tag{24}$$

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